

# AN ANALYTICAL METHOD TO ELIMINATE THE REDUNDANT PARAMETERS IN ROBOT CALIBRATION

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## Abstract

Model based error compensation of a robotic manipulator, also known as robot calibration, requires the identification of its generalized errors. These errors are found from measured data and used to predict, and compensate for, the end-point errors as a function of configuration. However, the generalized error formulation introduces redundant parameters, often non-intuitive, that may compromise the robustness of the calibration. The existing numerical methods to eliminate such errors are formulated on a case-by-case basis. In this paper, the general analytical expressions and physical interpretation of the redundant parameters are developed for any serial link manipulator, expressed through its Denavit-Hartenberg parameters. These expressions are used to eliminate the redundant parameters from the error model of any manipulator prior to the identification process, allowing for systematic robot calibration with improved accuracy.

## 1 Introduction

Robot calibration is a process to enhance robot position accuracy using compensation software. It involves identifying a more accurate functional relationship between the joint transducer readings and the workspace position of the end-effector [9]. The process requires the identification of the manipulator generalized errors from calibration measurements. Generalized errors characterize the relative position and orientation of frames defined at the manipulator links. These errors are found from measured data and used to predict, and compensate for, the end-point errors as a function of configuration.

Since some generalized errors result in end-effector errors in same direction, it is not possible to distinguish the amount of the error contributed by each generalized error. As a result, only linear combinations of the generalized errors can be identified, independently of the identification method used. All linear combinations must be eliminated from the error model prior to the identification process, otherwise the robustness of the calibration may be compromised [4, 6].

To eliminate redundant error parameters, a number of coordinate system representations have been considered. The four-parameter representations (such as the Denavit-Hartenberg representation) are attractive since they are the minimal parameter set required to locate the

reference frames of the joints [9]. Such representation reduces the number of error combinations to be found, however the redundant parameters are not necessarily eliminated. In addition, the Denavit-Hartenberg (D.H.) representation does not model some of the generalized errors (see section 3.5). The entire calibration can be compromised if such errors are significant. Also, the D.H. representation is ill-conditioned when neighboring joint axes are nearly parallel. Incorporating Hayati's proposed modification to the D.H. parameterization [2] eliminates the ill-conditioning problem, however it has a singularity when axes are nearly perpendicular [3]. Some authors have proposed the use of a five-parameter representation [5], however this parameterization has a sensitivity problem when neighboring coordinate origins are close together [12].

Many papers have abandoned the D.H. representation of the errors, treating the general case of two coordinate systems related by six parameters [3]. The six-parameter representation of the errors, called generalized error model, does not present the sensitivity problems of the D.H. representation. However, it has the disadvantage of increased redundancy [3]. Numerical methods have been proposed to eliminate redundant errors [1, 8, 10], however up to now they must be formulated in a case-by-case basis [4].

In this paper the analytical expressions and physical interpretation of the linear combinations of the generalized errors are developed for any serial link manipulator. The six-parameter representation is used to define the errors, and the linear combination coefficients are expressed through the robot's D.H. parameters. The error combinations using the D.H. four-parameter error representation are also derived from the general expressions. A non-singular form of the Identification Jacobian matrix is then obtained using these expressions, allowing for systematic calibration with improved accuracy of any serial link manipulator.

## 2 Analytical Background

### 2.1 Model Based Error Compensation

Physical errors, such as machining tolerances, assembly errors and elastic deformation, cause the geometric properties of a manipulator to be different from their ideal values. As a result, the frames defined at the manipulator joints are slightly displaced from their

expected, ideal locations, creating significant end-effector errors. As seen in Figure 1, the position and orientation of a frame  $F_i^{\text{real}}$  with respect to its ideal location  $F_i^{\text{ideal}}$  is represented by a 4x4 homogeneous matrix  $E_i$ . The translational part of matrix  $E_i$  is composed of 3 coordinates  $i_1$ ,  $i_2$  and  $i_3$  (along the X, Y and Z axes respectively, defined using the Denavit-Hartenberg representation). The rotation part of matrix  $E_i$  is the result of the product of three consecutive rotations  $i_4$ ,  $i_5$  and  $i_6$  around the Y, Z and X axes respectively (see Figure 2). These are the Euler angles of  $F_i^{\text{real}}$  with respect to  $F_i^{\text{ideal}}$ , representing spin (yaw), roll, and pitch, respectively. The 6 parameters  $i_1$ ,  $i_2$ ,  $i_3$ ,  $i_4$ ,  $i_5$  and  $i_6$  are called generalized error parameters, which can be a function of the system geometry and joint variables. For an  $n^{\text{th}}$  degree of freedom manipulator, there are  $6(n+1)$  generalized errors which can be written in vector form as  $\mathbf{e} = [\dots, i_1, i_2, i_3, i_4, i_5, i_6, \dots]$ , with  $i$  ranging from 0 to  $n$ . Here it is assumed that both the manipulator and the location of its base are being calibrated. If only the manipulator is being calibrated, then the error matrix  $E_0$  (which models the generalized errors of the robot base location) is not considered, reducing the number of generalized errors to  $6n$ .

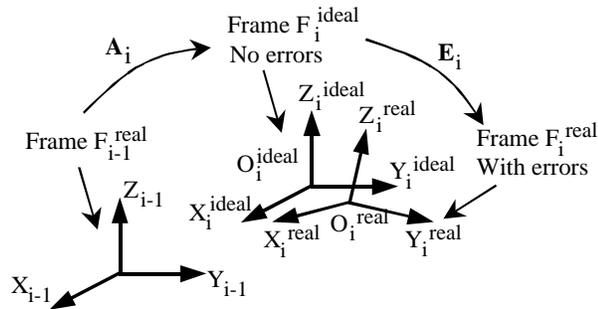


Figure 1 – Frame Translation and Rotation Due to Errors for  $i^{\text{th}}$  Link

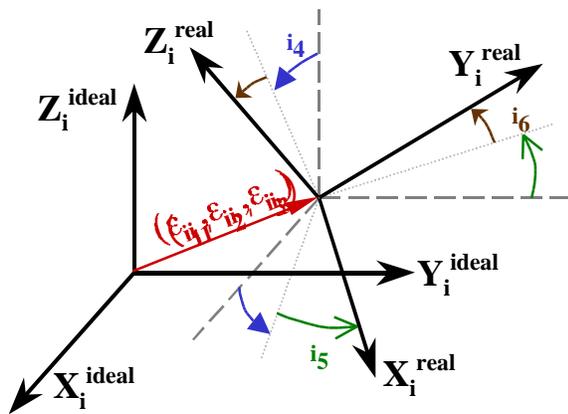


Figure 2 – Definition of the Translational and Rotational Generalized Errors for  $i^{\text{th}}$  Link

The end-effector position and orientation error  $\mathbf{X}$  is defined as the 6x1 vector that represents the difference between the real position and orientation of the end-effector and the ideal or desired one:

$$\mathbf{X} = \mathbf{X}^{\text{real}} - \mathbf{X}^{\text{ideal}} \quad (1)$$

where  $\mathbf{X}^{\text{real}}$  and  $\mathbf{X}^{\text{ideal}}$  are the 6x1 vectors composed of the three positions and three orientations of the end-effector reference frame in the inertial reference system for the real and ideal cases, respectively. Since the generalized errors are small,  $\mathbf{X}$  can be calculated by the following linear equation in :

$$\mathbf{X} = \mathbf{J}_e \quad (2)$$

where  $\mathbf{J}_e$  is the  $6 \times 6(n+1)$  Jacobian matrix of the end-effector error  $\mathbf{X}$  with respect to the elements of the generalized error vector  $\mathbf{e}$ , also known as Identification Jacobian matrix. Also, a first order approximation can then be applied to the trigonometric functions and products of the generalized errors. After the first order approximation, matrix  $E_i$  has the form:

$$\mathbf{E}_i = \begin{bmatrix} 1 & -i_5 & i_4 & i_1 \\ i_5 & 1 & -i_6 & i_2 \\ -i_4 & i_6 & 1 & i_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3)$$

If the generalized errors,  $\mathbf{e}$ , are known then the end-effector position and orientation error can be calculated using Equation (2). The method to obtain  $\mathbf{e}$  is explained in the next section.

## 2.2 Identification of the Generalized Errors

The identification method to calculate  $\mathbf{e}$  is based on the assumption that some components of vector  $\mathbf{X}$  can be measured at a finite number of different manipulator configurations. However, since position coordinates are much easier to measure in practice than orientations, in many cases only the three position coordinates of  $\mathbf{X}$  are measured. In this case, twice the number of measurements is required to obtain the same calibration accuracy.

Assuming that all 6 components of  $\mathbf{X}$  can be measured, for an  $n^{\text{th}}$  degree of freedom manipulator, its  $6(n+1)$  generalized errors  $\mathbf{e}$  can be calculated by measuring  $\mathbf{X}$  at  $m$  different configurations, defined as  $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_m$ , and then writing Equation (2)  $m$  times:

$$\mathbf{X}_t = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \dots \\ \mathbf{X}_m \end{bmatrix} = \begin{bmatrix} \mathbf{J}_e(\mathbf{q}_1) \\ \mathbf{J}_e(\mathbf{q}_2) \\ \dots \\ \mathbf{J}_e(\mathbf{q}_m) \end{bmatrix} = \mathbf{J}_t \quad (4)$$

where  $\mathbf{X}_t$  is the  $m \times 1$  vector formed by all measured vectors  $\mathbf{X}$  at the  $m$  different configurations and  $\mathbf{J}_t$  is the  $6m \times 6(n+1)$  Total Identification Jacobian matrix formed by the  $m$  Identification Jacobian matrices  $\mathbf{J}_e$  at the  $m$

configurations. To compensate for the effects of measurement noise, the number of measurements  $m$  is in general much larger than  $n$ .

If the generalized errors are constant, then a unique least-squares estimate  $\hat{\mathbf{x}}$  can be calculated by:

$$\hat{\mathbf{x}} = (\mathbf{J}_t^T \mathbf{J}_t)^{-1} \mathbf{J}_t^T \mathbf{X}_t \quad (5)$$

If the Identification Jacobian matrix  $\mathbf{J}_e$  contains linear dependent columns, then Equation (5) can give estimates with poor accuracy due to singularities. This occurs if there is redundancy in the error model, which is always true for the six-parameter representation. In this case, it is not possible to distinguish the amount of the error contributed by each generalized error  $\mathbf{j}_i$  by measuring the end-effector position and orientation, and only linear combinations of  $\mathbf{j}_i$  can be obtained. If only the position coordinates of vector  $\mathbf{X}$  are measured, then additional linear combinations may be present.

If the columns of  $\mathbf{J}_e$  are reduced to a linear independent set, then the non-singular form of the Identification Jacobian matrix, called  $\mathbf{G}_e$ , is obtained. In this case, the generalized errors must be grouped into a smaller independent set, in accordance with the columns of the submatrix  $\mathbf{G}_e$ . If  $\mathbf{J}_e$  is replaced by its submatrix  $\mathbf{G}_e$  in Equation (4), then Equation (5) can be applied resulting in estimates with improved accuracy. The next section shows how to obtain the non-singular Identification Jacobian matrix  $\mathbf{G}_e$ .

### 3 Eliminating the Redundant Errors

To perform robot calibration with improved accuracy, redundant errors must be eliminated from the error model prior to the identification process. Section 3.1 presents the linear combinations of the columns of the Identification Jacobian matrix and the method to obtain the non-singular Identification Jacobian matrix. Section 3.2 discusses the physical interpretation of each linear combination. Section 3.3 presents the additional linear combinations introduced when only the end-effector position is measured. Section 3.4 shows the number of independent error parameters for any serial link manipulator. Section 3.5 extends the results obtained using the six-parameter representation to the Denavit-Hartenberg error parameterization. It also shows that the D.H. representation of the errors does not model some of the generalized errors, which can affect the identification process, compromising the calibration.

#### 3.1 Linear Combinations of the Columns of the Identification Jacobian matrix

In this section, the linear combinations of the columns of the Identification Jacobian matrix  $\mathbf{J}_e$  are presented. Defining  $\mathbf{J}_i$  as the column of  $\mathbf{J}_e$  associated to the generalized error  $\mathbf{j}_i$ , then Equation (2) is rewritten as

$$\mathbf{X} = [\mathbf{J}_{11}, \dots, \mathbf{J}_{11}, \mathbf{J}_{12}, \mathbf{J}_{13}, \mathbf{J}_{14}, \mathbf{J}_{15}, \mathbf{J}_{16}, \dots, \mathbf{J}_{n6}]$$

$$[ \mathbf{j}_1, \dots, \mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3, \mathbf{j}_4, \mathbf{j}_5, \mathbf{j}_6, \dots, \mathbf{j}_6 ]^T \quad (6)$$

For each link  $i$  ( $1 \leq i \leq n$ ), the following linear combinations are always valid (see Appendix A for proof):

$$\mathbf{J}_{(i-1)3} \dots \sin \mathbf{j}_{i2} + \cos \mathbf{j}_{i3} \mathbf{J}_{i3} \quad (7)$$

$$\mathbf{J}_{(i-1)5} \dots a_i \cos \mathbf{j}_{i2} - a_i \sin \mathbf{j}_{i3} + \sin \mathbf{j}_{i4} + \cos \mathbf{j}_{i5} \mathbf{J}_{i5} \quad (8)$$

where the manipulator link parameters are defined using the D.H. representation: link lengths  $a_i$ , joint offsets  $d_i$ , joint angles  $\mathbf{j}_i$ , and skew angles  $\mathbf{j}_i$ . If joint  $i$  is prismatic, then two additional combinations of the columns of  $\mathbf{J}_e$  are found:

$$\mathbf{J}_{(i-1)1} \dots \mathbf{J}_{i1} \quad (9)$$

$$\mathbf{J}_{(i-1)2} \dots \cos \mathbf{j}_{i2} - \sin \mathbf{j}_{i3} \mathbf{J}_{i3} \quad (10)$$

The linear combinations shown above are always present, independently of the values of  $a_i$  and  $\mathbf{j}_i$ , even for degenerate cases (such as  $a_i=0$ ). As shown in Appendix A, if the full pose of the end-effector (both position and orientation) is measured, then Equations (7)-(10) are the only linear combinations for link  $i$ .

To obtain the non-singular Identification Jacobian matrix  $\mathbf{G}_e$ , columns  $\mathbf{J}_{(i-1)3}$  and  $\mathbf{J}_{(i-1)5}$  must be eliminated from the matrix  $\mathbf{J}_e$  for all values of  $i$  ( $1 \leq i \leq n$ ). Columns  $\mathbf{J}_{(i-1)1}$  and  $\mathbf{J}_{(i-1)2}$  must also be eliminated if and only if joint  $i$  is prismatic. For an  $n$ -d.o.f. manipulator with  $r$  rotary joints and  $p$  ( $=n-r$ ) prismatic joints,  $2r+4p$  columns are eliminated from the Identification Jacobian  $\mathbf{J}_e$  to form its submatrix  $\mathbf{G}_e$ . This means that exactly  $2r+4p$  generalized errors are not obtainable by measuring the end-effector pose.

If the vector  $\mathbf{x}^*$  containing the independent errors is constant, then the matrix  $\mathbf{G}_e$  can be used to replace  $\mathbf{J}_e$  in Equation (4), and Equation (5) is applied to calculate the estimate of the independent generalized errors  $\mathbf{x}^*$ , completing the identification process. Note that the independent errors  $\mathbf{x}^*$  are a subset of the generalized errors  $\mathbf{x}$ . However, if non-geometric factors are considered (e.g. link compliance, gear eccentricity), then it is necessary to further model the parameters of  $\mathbf{x}^*$  as a function of the system configuration prior to the identification process [7].

By definition, unobtainable parameters do not affect the end-effector error, resulting in the identity

$$\mathbf{X} = \mathbf{J}_e \mathbf{G}_e \mathbf{x}^* \quad (11)$$

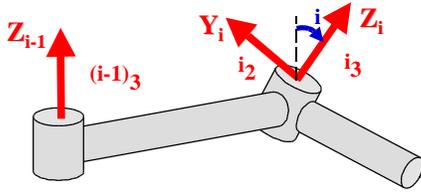
Using the above identity and the linear combinations of the columns of  $\mathbf{J}_e$  from Equations (7)-(10), it is possible to obtain all relationships between the generalized error set  $\mathbf{x}$  and its independent subset  $\mathbf{x}^*$ . These expressions are presented in Appendix A.

#### 3.2 Physical Interpretation of the Linear Combinations

In this section the physical interpretation of Equations (7)-(10) is presented. Each equation associates a

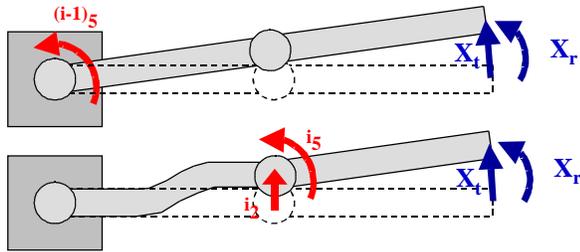
generalized error from link  $i-1$  with a combination of errors from link  $i$  that result in end-effector errors in same magnitude and direction. Since it is not possible to distinguish the amount of the error contributed by each generalized error, these errors associated with link  $i-1$  are indistinguishable.

Equation (7) reflects the fact that the translational error along the Z-axis of frame  $i-1$  has the same effect as a combination of the translational errors along the Y and Z axes of frame  $i$  (see Figure 3). This relation is easily explained by the fact that the skew angle  $\alpha_i$  between the axes of joints  $i-1$  and  $i$  is constant.



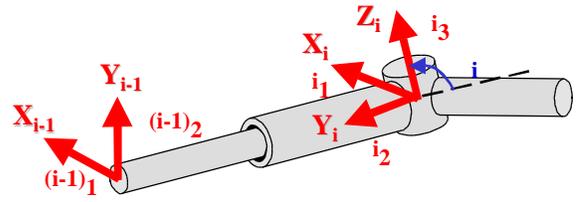
**Figure 3 – Linear Combination of Translational Generalized Errors**

Equation (8) states that the rotational error along the Z-axis of frame  $i-1$  has the same effect as a combination of the rotational and translational errors along the Y and Z axes of frame  $i$ . For simplicity, a planar manipulator is used to explain this combination (see Figure 4). The top figure shows the end-effector translational and rotational errors  $X_r$  and  $Y_r$  caused by the rotational generalized error  $(i-1)_5$  of frame  $i-1$ . The bottom figure shows that the same end-effector errors can be reproduced by a specific combination of the translational error  $i_2$  and the rotational error  $i_5$  of frame  $i$ . In this case, it is required that  $i_2 = (i-1)_5 a_i$  and  $i_5 = (i-1)_5$  to obtain the same end-effector errors (see the relationship between  $*$  and  $\cdot$  in Appendix A).



**Figure 4 – Error Combinations Resulting in Same End-Effector Errors**

If joint  $i$  is prismatic, then Equations (9) and (10) are also valid. These combinations simply state that the effects of the generalized errors along the X and Y axes of frame  $i-1$  can always be reproduced by a combination of the three translational generalized errors of frame  $i$  (see Figure 5). This is always true for prismatic joints, since such joints only move along the Z-axis of frame  $i-1$  (using the D.H. frame definition).



**Figure 5 – Linear Combinations of Generalized Errors in Prismatic Joints**

### 3.3 Partial Measurement of End-Effector Pose

The linear combinations of the columns of the Identification Jacobian matrix  $J_e$  shown in Equations (7)-(10) are obtained when both position and orientation of the end-effector are considered. In the case where only the end-effector position is measured, its orientation can take any value, resulting in additional linear combinations. In this case, the three last columns of  $J_e$  are zero vectors (see Appendix A):

$$J_{n_4} \dots J_{n_5} \dots J_{n_6} \dots 0 \quad (12)$$

Equation (12) means, as expected, that the three rotational errors of the end-effector frame  $n_4$ ,  $n_5$  and  $n_6$  do not influence the end-effector position (they only affect the orientation, which is not being measured). As a result, these generalized errors are not obtainable.

If the last joint is prismatic, then no further linear combinations are found. However, if the last joint is revolute and its link length  $a_n$  is zero, then three more linear combinations are present (see Appendix A):

$$J_{n-1_4} \dots d_n J_{n-1_1}, J_{n-1_6} \dots -d_n J_{n-1_2}, J_{n-1_5} \dots 0 \quad (13)$$

meaning that the effects of  $n-1_4$  and  $n-1_6$  cannot be distinguished from the ones caused by  $n-1_1$  and  $n-1_2$ , and also the generalized error  $n-1_5$  is not obtainable. If both link length  $a_n$  and joint offset  $d_n$  are zero, then the origin of frames  $n-1$  and  $n$  coincide at the end-effector position. In this case, Equations (12) and (13) can be recursively applied to frames  $n-1$ ,  $n-2$ , and so on, as long as the origin of these frames all lie at the end-effector position. See Appendix A for more details.

### 3.4 Number of Independent Generalized Errors

As a corollary of Equations (7)-(13), the number of independent generalized errors for a generic serial link manipulator can be calculated. Table 1 shows the number of generalized errors  $N$ , the number of linear dependencies  $D$ , and the number of independent generalized errors  $I (=N-D)$  for both robot calibration (without modeling its base frame errors) and robot+base location calibration.

	Robot+Base Calibration	Robot Calibration
$N$	$6(n+1)$	$6n$

<b>D</b>	$2r + 4p + k$	$2r' + 4p' + k$
<b>I</b>	$6(n+1) - (2r + 4p + k)$	$6n - (2r' + 4p' + k)$

**Table 1 – Number of Independent Generalized Errors**

where

$n$  : # of joints in the manipulator

$r$  ;  $r'$  : # of revolute joints including/excluding joint 1

$p$  ;  $p'$  : # of prismatic joints including/excluding joint 1

0 if measuring end-effector position and orientation

3 if only measuring end-effector position and either the last joint is prismatic or  $q = 0$

$k$  :  $3 + 2q$  if only measuring end-effector position, the last  $q$  joints are revolute, and

$a_{n-q+1} \ a_{n-q+2} \ a_{n-q+3} \ \dots \ a_n \ 0$

and  $d_{n-q+2} \ d_{n-q+3} \ \dots \ d_n \ 0$

### 3.5 Extension to Four-Parameter Error Representations

The six-parameter representation of the errors is used above to obtain all linear combinations of the generalized errors. If a four-parameter representation is chosen for the identification process, the previous results can still be applied to eliminate the redundant parameters, through an adaptation of Equations (7)-(10). The extension of the results to four-parameter error representations is easily accomplished because such parameterizations are a subset of the generalized error representation.

The four Denavit-Hartenberg error parameters of link  $i$  are exactly the rotational and translational errors along the Z-axis of frame  $i-1$  and the X-axis of frame  $i$  of the six-parameter representation. Namely, the errors along the link lengths  $a_i$ , skew angles  $\alpha_i$ , joint offsets  $d_i$ , and joint angle offsets  $\theta_i$  of link  $i$  are respectively mapped to the generalized errors  $i_1$ ,  $i_6$ ,  $(i-1)_3$ , and  $(i-1)_5$ . This implies that the translational and rotational errors along the Y-axis of every frame,  $i_2$  and  $i_4$ , are not modeled when the D.H. representation is used. However, both errors  $i_2$  and  $i_4$  are always obtainable if the full end-effector pose is measured (except for  $i_2$  if joint  $i+1$  is prismatic, as seen in section 3.1), meaning that the entire calibration can be compromised if the manipulator has significant errors in those directions. Hayati's modification of the D.H. representation [2] only partially solves this problem, because it introduces an angular alignment parameter that models the Y-axis rotation error  $i_4$ , but not the translation error  $i_2$ .

When using the D.H. error representation, the column vectors  $\mathbf{J}_{i_2}$  and  $\mathbf{J}_{i_4}$  are not present in the Identification Jacobian, because the generalized errors  $i_2$  and  $i_4$  are not modeled. So, Equations (7)-(10) can only be applied to the D.H. representation if both  $\mathbf{J}_{i_2}$  and  $\mathbf{J}_{i_4}$  are not present in the linear combination. This is never true for Equations (8) and (10), but Equation (7) can be applied if  $\sin(\alpha_i)$  is null, due to parallel joints. In this case, the following linear combinations of the D.H. error parameters are valid:

$$\begin{aligned} i_3^* &= i_3 + i_{-1} & ? d_{i+1}^* &=? d_{i+1} + ?d_i \\ i_5^* &= i_5 + i_{-1} & ??_{i+1}^* &=?_{i+1} + ??_i \end{aligned} \quad (14)$$

Finally, if joint  $i$  is prismatic, then Equation (9) results in one additional linear combination

$$i_1^* = i_1 + i_{-1} \quad ?a_i^* = ?a_i + ?a_{i-1} \quad (15)$$

As seen in Equations (14) and (15), even though the D.H. representation results in fewer linear combinations, redundant parameters are still present.

In summary, the D.H. representation of the errors, besides being ill-conditioned, does not model some of the physical errors and still presents linear combinations that need to be eliminated. Since the redundancy of the six-parameter error representation can be eliminated with the method described in this paper, the use of such parameterization in robot calibration is recommended.

## 4 Conclusions

This paper presents a general analytical method to eliminate the redundant error parameters in robot calibration. These errors, often non-intuitive, must be eliminated from the error model prior to the identification process, otherwise the robustness of the calibration can be compromised. The analytical expressions and physical interpretation of the linear combinations present in the generalized error parameterization are developed. The non-redundant form of the Identification Jacobian matrix is then obtained using these expressions, allowing for the systematic calibration with improved accuracy of any serial link manipulator.

## Acknowledgments

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## Appendix A – Linear Dependency Calculations

This appendix contains the proof of the linear combination expressions of the columns of the Identification Jacobian matrix  $\mathbf{J}_e$ . These combinations are obtained from the symbolic form of  $\mathbf{J}_e$ , expressed through the manipulator's Denavit-Hartenberg parameters. It is shown that the general expressions can be broken down into combinations of the columns associated with each pair of consecutive links. This in turn allows for great simplification of the proof.

Define the position and orientation of a reference frame  $F_i$  with respect to the previous reference frame  $F_{i-1}$  as a 4x4 matrix  $\mathbf{A}_i$  using the D.H. parameters:

$$\mathbf{A}_i = \begin{bmatrix} \cos \theta_i & -\sin \theta_i \cos \alpha_i & \sin \theta_i \sin \alpha_i & a_i \cos \theta_i \\ \sin \theta_i & \cos \theta_i \cos \alpha_i & -\cos \theta_i \sin \alpha_i & a_i \sin \theta_i \\ 0 & \sin \alpha_i & \cos \alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (16)$$

When the generalized errors defined in section 2.1 are considered in the model, the manipulator loop closure equation takes the form:

$$\mathbf{A}_{LC} = \mathbf{E}_0 \mathbf{A}_1 \mathbf{E}_1 \mathbf{A}_2 \mathbf{E}_2 \dots \mathbf{A}_n \mathbf{E}_n \quad (17)$$

where  $\mathbf{A}_{LC}$  is a 4x4 homogeneous matrix that describes the position and orientation of the end-effector frame  $F_n$  with respect to the inertial reference frame  $F_0$ .

The Identification Jacobian matrix  $\mathbf{J}_e$  is determined by taking the derivative of the loop closure matrix  $\mathbf{A}_{LC}$  with respect to each generalized error  $\theta_{ij}$ , resulting in 4x4 sensitivity matrices  $\mathbf{L}_{ij}$

$$\mathbf{L}_{ij} = \left. \frac{\partial (\mathbf{E}_0 \mathbf{A}_1 \mathbf{E}_1 \mathbf{A}_2 \mathbf{E}_2 \dots \mathbf{A}_n \mathbf{E}_n)}{\partial \theta_{ij}} \right|_{\theta_m = 0, k \neq i \text{ or } j} \quad (18)$$

Clearly, the linear combinations of the columns  $\mathbf{J}_{ij}$  of the Identification Jacobian matrix (defined in section 3.1) are the same as the ones of the  $\mathbf{L}_{ij}$  matrices. Since  $\mathbf{E}_i$  is the only matrix that depends on the generalized error  $\theta_{ij}$ , then  $\mathbf{L}_{ij}$  can be simplified to

$$\mathbf{L}_{ij} = \mathbf{A}_1 \dots \mathbf{A}_{i-2} \mathbf{M}_{ij} \mathbf{A}_{i+1} \dots \mathbf{A}_n, \quad \mathbf{M}_{ij} = \mathbf{A}_{i-1} \mathbf{A}_i \frac{\partial \mathbf{E}_i}{\partial \theta_{ij}} \quad (19)$$

Applying the same result for joint i-1, then

$$\mathbf{L}_{(i-1),j} = \mathbf{A}_1 \dots \mathbf{A}_{i-2} \mathbf{M}_{(i-1),j} \mathbf{A}_{i+1} \dots \mathbf{A}_n, \quad \mathbf{M}_{(i-1),j} = \mathbf{A}_{i-1} \frac{\partial \mathbf{E}_{i-1}}{\partial \theta_{(i-1),j}} \mathbf{A}_i \quad (20)$$

Since both products  $\mathbf{A}_1 \dots \mathbf{A}_{i-2}$  and  $\mathbf{A}_{i+1} \dots \mathbf{A}_n$  do not depend on the coordinates of frames i-1 and i, Equations (19) and (20) show that the linear combinations of the  $\mathbf{L}_{ij}$  matrices are also the same as the ones of the  $\mathbf{M}_{ij}$  matrices. Hence, it is not necessary to develop the lengthy Equations (18)-(20), since the desired linear combinations can be obtained from the much simpler  $\mathbf{M}_{ij}$  matrices. In addition, all linear combination expressions can then be broken down into expressions involving the generalized errors of each two consecutive links, since the considered system is a serial link manipulator.

Consider the following general expression for the linear combinations associated with two consecutive links, represented by unknown the coefficients  $c_1, \dots, c_{12}$ :

$$\begin{aligned} & c_1 J_{i-1} + c_2 J_{i-1} + c_3 J_{i-1} + c_4 J_{i-1} + c_5 J_{i-1} + c_6 J_{i-1} \\ & c_7 J_{i-1} + c_8 J_{i-1} + c_9 J_{i-1} + c_{10} J_{i-1} + c_{11} J_{i-1} + c_{12} J_{i-1} \end{aligned} \quad (21)$$

To determine the unknown coefficients, the column vectors  $\mathbf{J}_{ij}$  can be replaced by the matrices  $\mathbf{M}_{ij}$  in the above expression, since it was shown that both  $\mathbf{J}_{ij}$  and  $\mathbf{M}_{ij}$  sets have the same linear combinations. The  $\mathbf{M}_{ij}$  matrices are obtained by substituting Equations (3) and (16) into Equations (19) and (20). Equation (21) then results in

$$\begin{aligned} & c_4 (w_i \cos \theta_i) + c_5 (v_i \cos \theta_i - u_i \sin \theta_i) + c_6 (w_i \sin \theta_i) \\ & + c_{10} (u_i \sin \theta_i \sin \alpha_i - v_i \cos \theta_i \sin \alpha_i + w_i \cos \theta_i) \\ & - c_{11} (-u_i \sin \alpha_i \cos \theta_i + v_i \cos \alpha_i \cos \theta_i + w_i \sin \alpha_i) = 0 \\ & c_4 (-w_i \sin \theta_i \cos \alpha_i + u_i \sin \theta_i) - c_5 \cos \theta_i (v_i \sin \theta_i + u_i \cos \theta_i) \\ & + c_6 (w_i \cos \theta_i \cos \alpha_i - v_i \sin \theta_i) + c_{11} (u_i \cos \alpha_i + v_i \sin \alpha_i) \\ & + c_{12} (u_i \sin \theta_i \sin \alpha_i - v_i \cos \theta_i \sin \alpha_i + w_i \cos \theta_i) = 0 \\ & c_4 (w_i \sin \theta_i \sin \alpha_i + u_i \cos \theta_i) + c_5 \sin \theta_i (v_i \sin \theta_i + u_i \cos \theta_i) \\ & - c_6 (w_i \cos \theta_i \sin \alpha_i - v_i \cos \theta_i) - c_{10} (u_i \cos \alpha_i + v_i \sin \alpha_i) \\ & + c_{12} (-u_i \sin \alpha_i \cos \theta_i + v_i \cos \alpha_i \cos \theta_i + w_i \sin \alpha_i) = 0 \\ & c_1 u_i + c_2 v_i + c_3 w_i + c_4 (w_i a_i \cos \theta_i + u_i d_i) \\ & + c_5 (v_i a_i \cos \theta_i - u_i a_i \sin \theta_i) \\ & + c_6 (w_i a_i \sin \theta_i - v_i d_i) - c_7 (u_i \cos \theta_i + v_i \sin \theta_i) \\ & - c_8 (-u_i \sin \theta_i \cos \alpha_i + v_i \cos \alpha_i \cos \theta_i + w_i \sin \alpha_i) \\ & - c_9 (u_i \sin \theta_i \sin \alpha_i - v_i \cos \theta_i \sin \alpha_i + w_i \cos \theta_i) = 0 \end{aligned} \quad (22)$$

where

$$\begin{aligned} & u_i = \begin{bmatrix} \cos \theta_{i-1} & -\sin \theta_{i-1} \cos \alpha_{i-1} & \sin \theta_{i-1} \sin \alpha_{i-1} \\ \sin \theta_{i-1} & \cos \theta_{i-1} \cos \alpha_{i-1} & -\cos \theta_{i-1} \sin \alpha_{i-1} \\ 0 & \sin \alpha_{i-1} & \cos \alpha_{i-1} \\ 0 & 0 & 0 \end{bmatrix}, \quad v_i = \begin{bmatrix} \cos \theta_{i-1} & -\sin \theta_{i-1} \cos \alpha_{i-1} & \sin \theta_{i-1} \sin \alpha_{i-1} \\ \sin \theta_{i-1} & \cos \theta_{i-1} \cos \alpha_{i-1} & -\cos \theta_{i-1} \sin \alpha_{i-1} \\ 0 & \sin \alpha_{i-1} & \cos \alpha_{i-1} \\ 0 & 0 & 0 \end{bmatrix}, \quad w_i = \begin{bmatrix} \cos \theta_{i-1} & -\sin \theta_{i-1} \cos \alpha_{i-1} & \sin \theta_{i-1} \sin \alpha_{i-1} \\ \sin \theta_{i-1} & \cos \theta_{i-1} \cos \alpha_{i-1} & -\cos \theta_{i-1} \sin \alpha_{i-1} \\ 0 & \sin \alpha_{i-1} & \cos \alpha_{i-1} \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (23)$$

Solving Equation (22) results in

$$\begin{aligned} c_4 = c_6 = c_{12} = 0, \quad c_1 = c_7, \quad c_{10} = c_5 \sin \theta_i, \quad c_{11} = c_5 \cos \theta_i, \\ c_8 = c_3 \sin \theta_i + (c_2 + c_5 a_i) \cos \theta_i, \\ c_9 = c_3 \cos \theta_i - (c_2 + c_5 a_i) \sin \theta_i \end{aligned} \quad (24)$$

Then, substituting Equation (24) in (22) results in Equations (7) and (8). If joint i is revolute, then  $c_1 = c_2 = c_7 = 0$ , and no other combinations are present. However, if joint i is prismatic, then Equation (24) also results in Equations (9) and (10).

In the particular case where only the end-effector position is measured, only the last equation in (22) must hold, resulting in

$$\begin{aligned} c_1 = c_4 = c_6 = c_7, \quad c_8 = c_3 \sin \theta_n + (c_2 + c_5 a_n) \cos \theta_n, \\ c_9 = c_3 \cos \theta_n - (c_2 + c_5 a_n) \sin \theta_n \end{aligned} \quad (25)$$

Substituting Equation (25) in (21) results in

$$\begin{aligned} & c_3 J_{n-1} + c_5 J_{n-1} (c_3 \sin \theta_n + c_5 a_n \cos \theta_n) J_{n-1} + \\ & (c_3 \cos \theta_n - c_5 a_n \sin \theta_n) J_{n-1} + c_{10} J_{n-1} + c_{11} J_{n-1} + c_{12} J_{n-1} \end{aligned} \quad (26)$$

Equation (26) must hold for all values of  $c_3, c_5, c_{10}, c_{11}, c_{12}$ , resulting in Equation (12).

If joint n is prismatic, then  $c_4$  and  $c_6$  are always zero (even if  $a_n = 0$ ) and no other combinations are present. However, if joint n is revolute and the link length  $a_n$  is zero, then  $c_4$  and  $c_6$  are different than zero, and two other linear combinations are present, namely

$$\begin{aligned} c_1 + c_4 d_n = 0 \quad (-c_4 d_n) J_{n-1} + c_4 J_{n-1} = 0 \quad J_{n-1} = d_n J_{n-1} \\ c_2 - c_6 d_n = 0 \quad (c_6 d_n) J_{n-1} + c_6 J_{n-1} = 0 \quad J_{n-1} = -d_n J_{n-1} \end{aligned} \quad (27)$$

Equations (8) and (12) also imply that  $\mathbf{J}_{(n-1),5} = \mathbf{0}$  for this case. If the joint offset  $d_n$  is also zero, then reference frames n and n-1 have common origins at the manipulator end-effector, and Equations (8) and (13) imply that

$$\mathbf{J}_{n-4} \cdots \mathbf{J}_{n-5} \cdots \mathbf{J}_{n-6} \cdots \mathbf{0} \quad (28)$$

Equations (12) and (13) are then recursively applied for link n-1.

Once the linear combinations of the columns of the Jacobian matrix  $\mathbf{J}_e$  are calculated, the independent generalized error set is obtained. From Equation (7),

$$\begin{aligned} & \mathbf{J}_{i-1_3 \ i-1_3} + \mathbf{J}_{i_2 \ i_2} + \mathbf{J}_{i_3 \ i_3} \cdots \mathbf{J}_{i_2 \ (i_2 + i-1_3 \ \sin \ i)} + \\ & \mathbf{J}_{i_3 \ (i_3 + i-1_3 \ \cos \ i)} \cdots \mathbf{J}_{i_2 \ i_2}^* + \mathbf{J}_{i_3 \ i_3}^* \end{aligned} \quad (29)$$

showing that the generalized error  $i-1_3$  can be incorporated into  $i_2$  and  $i_3$  if joint i is revolute, resulting in the combined generalized errors  $i_2^*$  and  $i_3^*$ . Using Equations (7) and (8) and the approach described above, the combined generalized errors are obtained:

$$\begin{aligned} i_2^* &= i_2 + i-1_3 \sin \ i + i-1_5 \ a_1 \cos \ i \\ i_3^* &= i_3 + i-1_3 \cos \ i - i-1_5 \ a_1 \sin \ i \\ i_4^* &= i_4 + i-1_5 \sin \ i \\ i_5^* &= i_5 + i-1_5 \cos \ i \end{aligned} \quad (30)$$

which holds for both revolute and prismatic joints.

If joint i is prismatic, then Equations (9) and (10) are combined with Equation (30), resulting in

$$\begin{aligned} i_4^* &= i_4 + i-1_5 \\ i_2^* &= i_2 + i-1_5 \cos \ i + i-1_3 \sin \ i + i-1_5 \ a_1 \cos \ i \\ i_3^* &= i_3 - i-1_5 \sin \ i + i-1_3 \cos \ i - i-1_5 \ a_1 \sin \ i \\ i_4^* &= i_4 + i-1_6 \sin \ i \\ i_5^* &= i_5 + i-1_5 \cos \ i \end{aligned} \quad (31)$$

In the particular case where the end-effector orientation is not considered, Equation (12) implies that the generalized errors  $n_4$ ,  $n_5$  and  $n_6$  do not affect the end-effector measurements. Furthermore, if the last joint is revolute and  $a_n = 0$  then Equation (27) results in the combined generalized errors

$$\begin{aligned} n_{n-1}^* &= n_{n-1} + n_{n-1} \ d_n \\ n_{n-1}^* &= n_{n-1} - n_{n-1} \ d_n \end{aligned} \quad (32)$$

and also implies that  $n_{n-1_5}$  does not influence the end-effector position.

## References

[1] Everett, L.J., Suryohadiprojo, A.H., 1988, "A Study of Kinematic Models for Forward Calibration of Manipulators," *Proc. IEEE International Conference of Robotics and Automation*, Philadelphia, pp.798-800.

[2] Hayati, S.A., 1983, "Robot Arm Geometry Link Parameter Estimation," *Proc. 22nd IEEE Conference Decision and Control*, San Antonio, pp.1477-1483.

[3] Hollerbach, J., 1988, "A Survey of Kinematic Calibration," *Robotics Review*, Khatib O. et al editors, Cambridge, MA, MIT Press.

[4] Hollerbach, J.M., Wampler, C.W., 1996, "The Calibration Index and Taxonomy for Robot Kinematic Calibration Methods," *International Journal of Robotics Research*, **15**(6): pp.573-591.

[5] Hsu, T.-W., Everett, L.J., 1985, "Identification of the Kinematic Parameters of a Robot Manipulator for Positional Accuracy Improvement," *Proc. Comp.in Eng. Conference*, Boston, pp.263-267.

[6] Ikits, M., Hollerbach, J.M., 1997, "Kinematic Calibration Using a Plane Constraint," *IEEE International Conference on Robotics and Automation*, Albuquerque, New Mexico, pp.3191-3196.

[7] Meggiolaro, M., Mavroidis, C. and Dubowsky, S., "Identification and Compensation of Geometric and Elastic Errors in Large Manipulators: Application to a High Accuracy Medical Robot." *Proceedings of the 1998 ASME Design Engineering Technical Conference*, Atlanta, 1998.

[8] Press, W.H., Teukolsky, S.A., Vetterling, W.T., Flannery, B.P., 1992, *Numerical Recipes in C*. Cambridge, UK: Cambridge University Press.

[9] Roth, Z.S., Mooring, B.W., Ravani, B., 1987, "An Overview of Robot Calibration," *IEEE J. of Robotics and Automation*, **3**(5): pp.377-384.

[10] Schröer, K., 1993, *Theory of Kinematic Modelling and Numerical Procedures for Robot Calibration*. In Bernhardt, R., Albright, S.L. (eds.): *Robot Calibration*. London: Chapman & Hall, pp.157-196.

[11] Zhuang, H., Motaghedi, S.H., Roth, Z.S., 1999, "Robot Calibration with Planar Constraints," *Proc. IEEE International Conference of Robotics and Automation*, Detroit, Michigan, pp.805-810.

[12] Ziegert, J., Datsoris, P., 1988, "Basic Considerations for Robot Calibration," *Proc. IEEE International Conference on Robotics and Automation*, Philadelphia, pp.932-938.